

On the integrable gravity coupled to fermions

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Abstract

In the present paper we indicate an extension of the pure gravity inverse scattering integration technique (developed in [2]) to the case when fermions are present. With this extension the integrability of the maximal supergravity $N = 16$ in two space-time dimensions constructed in [1] is revisited. In addition to the results of the article [1] the spectral linear problem proposed in the present paper covers also the Dirac-like fermionic equations of motion and is free of the second order poles with respect to the spectral parameter. The procedure of constructing the exact super-solitonic solutions is outlined.

1 Introduction

The first integrable supergravity model appeared in 1987 by the paper of Hermann Nicolai [1]. In this article the integrable maximal $N = 16$ supergravity in two space-time dimensions has been established and corresponding Lax representation has been proposed. However, this construction has been not finished completely. First of all the Lax pair presented in [1] are not complete since it contains only bosonic part of the equations of motion. The first order differential equations of motion of the Dirac type for fermionic fields do not follow from this linear spectral problem and must be added by hands. Another undesirable point, which lead to some technical complications, is appearance in this linear spectral equations the poles of the second order with respect to the spectral parameter while the corresponding spectral problem in pure gravity has only simple poles.

The presence of the of the second order poles in the linear spectral problem proposed in [1] is evident directly from its defining form. The inability of this linear problem to cover the Dirac-like equations of motion for fermions also is evident from the fact that this Lax pair contains only the terms quadratic in the (anticommuting) fermionic fields. The equations of motion following from the Nicolai spectral linear problem consist of two groups. The first one represents the non-linear wave-type equations for the set of scalar fields with bosonic

sources containing only quadratic products of the fermions. The second group are the first order differential equations for these quadratic bosonic sources. The Dirac-like equations for fermions themselves does not result from this Lax representation and fermions can not be reconstructed from their quadratic combinations. However, it is interesting and essential that the aforementioned equations for the quadratic bosonic sources are the direct consequences of the physical Dirac-like equations for fermions although these last does not follow from the Nicolai spectral linear problem and stand outside of his Lax-type representation.

In the present paper we indicate in general the possible extension of the inverse scattering integration technique of the pure vacuum gravity developed in [2] to the case when (anticommuting) fermions are present. In the framework of this generalization we extend the Nicolai type of the Lax representation to the complete one (covering also the Dirac-like fermionic equations of motion) and liberate it from the second order poles. Also we outline the main steps of construction the exact super-solitonic solutions using such generalized approach.

2 The multidimensional version of the integrable gravity

It is known [2, 3] that in any space-time of dimension $n + 2$ with interval

$$ds^2 = g_{\alpha\beta}(x^0, x^1) dx^\alpha dx^\beta + g_{ik}(x^0, x^1) dy^i dy^k \quad (1)$$

(where $\alpha, \beta, \gamma, \dots = 0, 1$ and $i, k, l, \dots = 1, 2, \dots, n$) the Einstein equations are integrable since the equations for the metric coefficients $g_{ik}(x^0, x^1)$ follows from the Lax pair as its self-consistency conditions and the metric components $g_{\alpha\beta}(x^0, x^1)$ can be found by quadratures in terms of the known $g_{ik}(x^0, x^1)$. Without loss of generality the 2-dimensional block $g_{\alpha\beta}(x^0, x^1)$ can be chosen in conformal flat form:

$$g_{\alpha\beta} = \lambda^2 \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \text{diag}(\eta_{00}, \eta_{11}) = \text{diag}(1, -1). \quad (2)$$

It is convenient to introduce matrix \mathbf{G} (with entries G_{ik})¹ and represent the metric matrix \mathbf{g} (with entries g_{ik}) as

$$\mathbf{g} = \alpha^{2/n} \mathbf{G}, \quad \det \mathbf{G} = 1, \quad \det \mathbf{g} = \alpha^2. \quad (3)$$

Then the components $R_{\alpha i}$ of the Ricci tensor vanish identically and equations $R_{\alpha\beta} = 0$ and $R_{ik} = 0$ can be written in matrix form using the light-like variables ζ, η :

$$x^0 = \eta + \zeta, \quad x^1 = \eta - \zeta. \quad (4)$$

¹All multidimensional matrices (apart from the two-dimensional Dirac gamma-matrices γ^α) we designate by the bold letters. Tilde at the top of a matrix means transposition. The functions which depend also on the spectral parameter we designate by the letters with the hat on the top. The simple partial derivatives are denoted by comma.

Equations $R_{ik} = 0$ are:

$$\alpha_{,\zeta\eta} = 0 , \quad (5)$$

$$(\alpha \mathbf{G}^{-1} \mathbf{G}_{,\zeta})_{,\eta} + (\alpha \mathbf{G}^{-1} \mathbf{G}_{,\eta})_{,\zeta} = \mathbf{0}. \quad (6)$$

Equations $R_{\alpha\beta} = 0$ are equivalent to the system²:

$$\frac{f_{,\zeta}}{f} = \frac{\alpha}{4\alpha_{,\zeta}} \text{Tr} \left[(\mathbf{G}^{-1} \mathbf{G}_{,\zeta})^2 \right] , \quad (7)$$

$$\frac{f_{,\eta}}{f} = \frac{\alpha}{4\alpha_{,\eta}} \text{Tr} \left[(\mathbf{G}^{-1} \mathbf{G}_{,\eta})^2 \right] , \quad (8)$$

where

$$f = \frac{\alpha^{(n-1)/n} \lambda^2}{\alpha_{,\zeta} \alpha_{,\eta}} . \quad (9)$$

If equations (5) and (6) are solved³ then f can be found by quadratures from equations (7) and (8) [the self-consistency conditions of which are satisfied automatically if α and \mathbf{G} are the solutions of the equations (5) and (6)].

The crucial property of this ansatz of the Einstein theory is complete separation of the equation for the metric components $g_{ik}(x^0, x^1)$ from equations for the conformal factor λ^2 . Namely this phenomenon considerably facilitates the integration of gravitational equations for the metric (1).

The spectral linear problem associated with the main equation (6) we used in [2] contains the differentiation also with respect to the spectral parameter but this is an inessential technical point. Our original Lax representation for the equation (6) can be written also in the following equivalent form:

$$\hat{\mathbf{G}}^{-1} \hat{\mathbf{G}}_{,\zeta} = \frac{\alpha}{\alpha - s} \mathbf{G}^{-1} \mathbf{G}_{,\zeta} , \quad \hat{\mathbf{G}}^{-1} \hat{\mathbf{G}}_{,\eta} = \frac{\alpha}{\alpha + s} \mathbf{G}^{-1} \mathbf{G}_{,\eta} , \quad (10)$$

where $\hat{\mathbf{G}}(\zeta, \eta, s)$ depends on the complex spectral parameter s which depends on the coordinates ζ, η in accordance with differential equations:

$$\frac{s_{,\zeta}}{s} = \frac{2\alpha_{,\zeta}}{\alpha - s} , \quad \frac{s_{,\eta}}{s} = \frac{2\alpha_{,\eta}}{\alpha + s} . \quad (11)$$

The self-consistency requirement for the last equations is satisfied due to the condition (5). The solution of the equations (11) contains one arbitrary complex

²In fact the system $R_{\alpha\beta} = 0$ contains one more equation additional to the (7)-(8) which is of second order for f but it is the direct consequence of (5)-(8) and we can forget about it.

³The following important point should be stressed here. From the relations (3) follows that solution of equation (6) for the matrix \mathbf{G} should satisfy the restriction $\det \mathbf{G} = \mathbf{1}$. However, the application of the inverse scattering integration procedure to the equation (6), if taking this restriction from the outset, technically are a little bit awkward. More convenient approach is to solve (6) first ignoring any additional restriction for the $\det \mathbf{G}$ but at the end of calculation make the simple rescaling of the solution in order to get the necessary condition of the unit determinant. The trick is as follows. If we obtained the solution of the equation (6) with $\det \mathbf{G} \neq \mathbf{1}$ we can pass to the new "physical" matrix $\mathbf{G}_{(ph)}$ by the transformation $\mathbf{G}_{(ph)} = (\det \mathbf{G})^{-1/n} \mathbf{G}$. It is simple task to prove that the matrix $\mathbf{G}_{(ph)}$ is also a solution of the same equation (6) and simultaneously satisfy the condition $\det \mathbf{G}_{(ph)} = 1$.

constant w then parameter $s = s(\zeta, \eta, w)$ has one arbitrary degree of freedom independent of those due to the changing of coordinates. This means that in the integrability conditions for the pair (10) all terms containing the different powers of s must vanish separately. The matrix \mathbf{G} in the right hand side of (10) is a function on the two coordinates ζ and η only, that is it is treated as unknown "potential" independent on the parameter s . The function $\alpha(\zeta, \eta)$ which is a solution of the wave equation (5) should be considered in (10) as some given external field. The equation of interest (6) should result from the linear spectral system (10) as its self-consistency (integrability) conditions and it is easy to check that this is indeed the case.

For any regular at the point $s = 0$ solution $\hat{\mathbf{G}}(\zeta, \eta, s)$ of the Lax pair (10) the solution of the equation (6) for matrix $\mathbf{G}(\zeta, \eta)$ follows automatically from the relation:

$$\mathbf{G}(\zeta, \eta) = [\hat{\mathbf{G}}(\zeta, \eta, s)]_{s=0} . \quad (12)$$

In case of solitonic fields the procedure of integration of the spectral linear problem (10) for matrix $\hat{\mathbf{G}}(\zeta, \eta, s)$ consists of the following steps. First we need to have some background solutions $\alpha(\zeta, \eta)$ and $\mathbf{G}_0(\zeta, \eta)$ of the gravitational equations (5)-(6) and then to find from equations (10) the corresponding background spectral matrix $\hat{\mathbf{G}}_0(\zeta, \eta, s)$. After that we "dress" $\hat{\mathbf{G}}_0(\zeta, \eta, s)$ by the simplest meromorphic (having only isolated first-order poles with respect to the spectral parameter s) matrix $\hat{\mathbf{K}}(\zeta, \eta, s)$, that is we represent $\hat{\mathbf{G}}$ in the form $\hat{\mathbf{G}}(\zeta, \eta, s) = \hat{\mathbf{G}}_0(\zeta, \eta, s) \hat{\mathbf{K}}(\zeta, \eta, s)$. Substituting this form into equations (10) we can find the matrix $\hat{\mathbf{K}}(\zeta, \eta, s)$ in terms of the known background solution $\hat{\mathbf{G}}_0(\zeta, \eta, s)$ and given wave function $\alpha(\zeta, \eta)$ by pure algebraic procedure (namely this is the principal advantage of the method). The number of poles with respect to the spectral parameter s in matrix $\hat{\mathbf{K}}(\zeta, \eta, s)$ is the number of solitons we add to the background. Then the final solution of interest for $\mathbf{G}(\zeta, \eta)$ follows from the relation (12).

3 Generalization to the case when fermions are present

Generalization of the scheme described above to the case when a number of fermionic fields are present can be done by introducing the superspace with coordinates x^0, x^1, θ^a where $a = 1, 2, \dots, N$ and all θ^a are odd (anticommuting) variables. Now the spectral matrix $\hat{\mathbf{G}}(\zeta, \eta, s)$ which appeared in the equations (10) should be replaced by a supermatrix $\hat{\Psi}(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N, s)$ with even (commuting) entries and instead of the simple derivatives ∂_ζ and ∂_η in (10) we should use the following odd differential operators:

$$\begin{aligned} D_\zeta &= A^a \frac{\partial}{\partial \theta^a} - B_a \theta^a \frac{\partial}{\partial \zeta} , \\ D_\eta &= -E^a \frac{\partial}{\partial \theta^a} + F_a \theta^a \frac{\partial}{\partial \eta} , \end{aligned} \quad (13)$$

where A^a, E^a, B_a, F_a are even constants obeying the constraints

$$A^a F_a = 0, \quad E^a B_a = 0. \quad (14)$$

Under these constraints operators (13) anticommute:

$$D_\zeta D_\eta + D_\eta D_\zeta = 0, \quad (15)$$

and the superspace generalization of the Lax pair (10) becomes:

$$\hat{\Psi}^{-1} D_\zeta \hat{\Psi} = \frac{\alpha}{\alpha - s} \Psi^{-1} D_\zeta \Psi, \quad \hat{\Psi}^{-1} D_\eta \hat{\Psi} = \frac{\alpha}{\alpha + s} \Psi^{-1} D_\eta \Psi, \quad (16)$$

where $\alpha(x^0, x^1)$ still is a usual even function which does not depend on the θ -coordinates and satisfies the wave equation (5). The spectral parameter $s(x^0, x^1)$ also is even and does not depend on the θ -coordinates and follows from the differential equations (11) after function $\alpha(x^0, x^1)$ is fixed.

For any regular at the point $s = 0$ solution of equation (16) for $\hat{\Psi}$ the matrix of interest Ψ comes from the relation:

$$\Psi(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N) = [\hat{\Psi}(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N, s)]_{s=0} \quad (17)$$

The crucial point is that the self-consistency condition $D_\zeta (D_\eta \hat{\Psi}) + D_\eta (D_\zeta \hat{\Psi}) = \mathbf{0}$ of the generalized Lax pair (16) reduces only to one supermatrix equation [superspace analog of (6)]:

$$D_\zeta (\alpha \Psi^{-1} D_\eta \Psi) - D_\eta (\alpha \Psi^{-1} D_\zeta \Psi) = \mathbf{0}, \quad (18)$$

which in general is equivalent to the huge array of the interacting bosonic and fermionic fields which are represented by the coefficients in the expansion of $\Psi(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N)$ with respect to the θ -coordinates:

$$\begin{aligned} \Psi = \mathbf{J}(\zeta, \eta) [\mathbf{I} + \theta^a \Omega_a(\zeta, \eta) + \theta^a \theta^b \mathbf{H}_{ab}(\zeta, \eta) + \theta^a \theta^b \theta^c \Omega_{abc}(\zeta, \eta) \\ + \theta^a \theta^b \theta^c \theta^d \mathbf{H}_{abcd}(\zeta, \eta) + \dots], \end{aligned} \quad (19)$$

where \mathbf{I} is the unity, \mathbf{J} and all \mathbf{H} have even entries and all Ω consist of the odd entries. If we are able to find the solution of equations (16) for the spectral matrix $\hat{\Psi}(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N, s)$ (this is the main problem of the method, however, exactly solvable for the solitonic fields) then the solution of the equation (18) for matrix $\Psi(\zeta, \eta, \theta^1, \theta^2, \dots, \theta^N)$ follows automatically from the relation (17) together with all fields $\mathbf{J}(\zeta, \eta), \Omega_a(\zeta, \eta), \mathbf{H}_{ab}(\zeta, \eta), \dots$ of interest.

Of course, the point is whether these fields can have a physical application and in general this is an open question. However, in the present paper we demonstrate that at least the first non-trivial case when the spectral matrix $\hat{\Psi}$ depends only on two odd coordinates⁴ indeed has a physical interest since

⁴The model when spectral matrix does not depend on the odd coordinates at all is equivalent to the integrable pure gravity case we described in the previous section. The case when spectral matrix depends only on one such coordinate follows from the ansatz corresponding to two odd variables as particular case.

it contains the principal part of the integrable maximal $N = 16$ supergravity model proposed in [1]. Moreover the superspace Lax pair (16) is free from the technical nuisances we mentioned in Introduction, that is it does not contains the second order poles with respect to the spectral parameter and, as we will show, it covers all equations of motion including the Dirac-like equations for the fermionic fields.

If we have only two odd coordinates θ^1 and θ^2 then the matrix $\Psi(\zeta, \eta, \theta^1, \theta^2)$ has the structure:

$$\Psi = \mathbf{J}(\mathbf{I} + \theta^1 \mathbf{\Omega}_1 + \theta^2 \mathbf{\Omega}_2 + \theta^1 \theta^2 \mathbf{H}) , \quad (20)$$

$$\Psi^{-1} = [\mathbf{I} - \theta^1 \mathbf{\Omega}_1 - \theta^2 \mathbf{\Omega}_2 - \theta^1 \theta^2 \mathbf{H} - \theta^1 \theta^2 (\mathbf{\Omega}_1 \mathbf{\Omega}_2 - \mathbf{\Omega}_2 \mathbf{\Omega}_1)] \mathbf{J}^{-1} , \quad (21)$$

and, in accordance with constraints (14), operators (13) can be chosen as:

$$\begin{aligned} D_\zeta &= \frac{\partial}{\partial \theta^2} - \theta^2 \frac{\partial}{\partial \zeta} , \\ D_\eta &= -\frac{\partial}{\partial \theta^1} + \theta^1 \frac{\partial}{\partial \eta} . \end{aligned} \quad (22)$$

Substituting this into (18) one can show that this single equation in superspace is equivalent to the following system of differential equations⁵ in the space-time for the components $\mathbf{J}(\zeta, \eta)$, $\mathbf{\Omega}_1(\zeta, \eta)$, $\mathbf{\Omega}_2(\zeta, \eta)$, $\mathbf{H}(\zeta, \eta)$:

$$\begin{aligned} \frac{1}{\alpha} (\alpha \mathbf{J}^{-1} \mathbf{J}_{,\zeta})_{,\eta} + \frac{1}{\alpha} (\alpha \mathbf{J}^{-1} \mathbf{J}_{,\eta})_{,\zeta} &= \frac{1}{2} (\mathbf{J}^{-1} \mathbf{J}_{,\zeta} \mathbf{\Omega}_1^2 - \mathbf{\Omega}_1^2 \mathbf{J}^{-1} \mathbf{J}_{,\zeta}) \\ &+ \frac{1}{2} (\mathbf{J}^{-1} \mathbf{J}_{,\eta} \mathbf{\Omega}_2^2 - \mathbf{\Omega}_2^2 \mathbf{J}^{-1} \mathbf{J}_{,\eta}) , \end{aligned} \quad (23)$$

$$\frac{1}{\sqrt{\alpha}} (\sqrt{\alpha} \mathbf{\Omega}_1)_{,\zeta} + \frac{1}{2} (\mathbf{J}^{-1} \mathbf{J}_{,\zeta} \mathbf{\Omega}_1 - \mathbf{\Omega}_1 \mathbf{J}^{-1} \mathbf{J}_{,\zeta}) + \frac{1}{4} (\mathbf{\Omega}_2^2 \mathbf{\Omega}_1 - \mathbf{\Omega}_1 \mathbf{\Omega}_2^2) = 0 , \quad (24)$$

$$\frac{1}{\sqrt{\alpha}} (\sqrt{\alpha} \mathbf{\Omega}_2)_{,\eta} + \frac{1}{2} (\mathbf{J}^{-1} \mathbf{J}_{,\eta} \mathbf{\Omega}_2 - \mathbf{\Omega}_2 \mathbf{J}^{-1} \mathbf{J}_{,\eta}) + \frac{1}{4} (\mathbf{\Omega}_1^2 \mathbf{\Omega}_2 - \mathbf{\Omega}_2 \mathbf{\Omega}_1^2) = 0 , \quad (25)$$

$$\mathbf{H} = \frac{1}{2} (\mathbf{\Omega}_2 \mathbf{\Omega}_1 - \mathbf{\Omega}_1 \mathbf{\Omega}_2) . \quad (26)$$

As follows from the foregoing the equations (23)-(25) are integrable because they are the self-consistency conditions of the super Lax pair (16) but what is especially interesting they coincide with the principal part of the equations of motion of the Nicolai $N = 16$ supergravity model if the fields \mathbf{J} , $\mathbf{\Omega}_1$, $\mathbf{\Omega}_2$ take values in the non-compact $E_{8(8)}$ group. To show this coincidence explicitly it is

⁵Direct derivation of the first integrability condition of the Lax pair (16) gives it as $(\alpha \mathbf{J}^{-1} \mathbf{J}_{,\zeta} + \alpha \mathbf{\Omega}_2^2)_{,\eta} + (\alpha \mathbf{J}^{-1} \mathbf{J}_{,\eta} + \alpha \mathbf{\Omega}_1^2)_{,\zeta} = 0$, however using (24) and (25) it can be transformed to the form (23).

necessary to return in the system (23)-(25) to the Cartesian coordinates x^0, x^1 defined in (4) and to use instead of the metric \mathbf{J} the orthonormal frame \mathbf{V} :

$$\mathbf{J} = \mathbf{V}\tilde{\mathbf{V}}. \quad (27)$$

Instead of the matrices $\mathbf{\Omega}_1, \mathbf{\Omega}_2$ it is convenient to use their similar images $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$ with respect to the frame \mathbf{V} :

$$\mathbf{\Omega}_1 = \tilde{\mathbf{V}}^{-1}\mathbf{\Lambda}_1\tilde{\mathbf{V}}, \quad \mathbf{\Omega}_2 = \tilde{\mathbf{V}}^{-1}\mathbf{\Lambda}_2\tilde{\mathbf{V}}. \quad (28)$$

The frame current $\mathbf{V}^{-1}\mathbf{V}_{,\alpha}$ can be decomposed into antisymmetric and symmetric parts:

$$\mathbf{V}^{-1}\mathbf{V}_{,\alpha} = \mathbf{Q}_\alpha + \mathbf{P}_\alpha, \quad (29)$$

where matrices \mathbf{Q}_α are antisymmetric and matrices \mathbf{P}_α symmetric. From (27) and (29) we obtain the following useful expression for the metric current $\mathbf{J}^{-1}\mathbf{J}_{,\alpha}$:

$$\mathbf{J}^{-1}\mathbf{J}_{,\alpha} = 2\tilde{\mathbf{V}}^{-1}\mathbf{P}_\alpha\tilde{\mathbf{V}} \quad (30)$$

Now it is easy to see that equations (24) and (25) have a natural Dirac spinorial structure in two-dimensional Minkowski space-time with metric $\eta_{\alpha\beta}$ defined by (2) and with the following gamma-matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_0 = \gamma^0, \quad \gamma_1 = -\gamma^1. \quad (31)$$

If we introduce the matrix-generalized two-component Dirac-like spinor Φ

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \end{pmatrix}, \quad (32)$$

in which the components Φ_1 and Φ_2 are internal matrices of arbitrary size (their internal matrix structure has no relation to the two-dimensional Dirac algebra in two-dimensional space-time), then in terms of such spinor and frame current components $\mathbf{Q}_\alpha, \mathbf{P}_\alpha$ equations (23)-(25) take the form:

$$\eta^{\alpha\beta} \left[\frac{1}{\alpha} (\alpha \mathbf{P}_\alpha)_{,\beta} + \mathbf{Q}_\alpha \mathbf{P}_\beta - \mathbf{P}_\beta \mathbf{Q}_\alpha \right] = \frac{1}{8} \mathbf{P}_\alpha (\bar{\Phi} \gamma^\alpha \Phi) - \frac{1}{8} (\bar{\Phi} \gamma^\alpha \Phi) \mathbf{P}_\alpha, \quad (33)$$

$$\begin{aligned} \frac{1}{\sqrt{\alpha}} \gamma^\alpha (\sqrt{\alpha} \Phi)_{,\alpha} + \mathbf{Q}_\alpha (\gamma^\alpha \Phi) - (\gamma^\alpha \Phi) \mathbf{Q}_\alpha &= \frac{1}{16} (\gamma_\alpha \Phi) (\bar{\Phi} \gamma^\alpha \Phi) \\ &\quad - \frac{1}{16} (\bar{\Phi} \gamma^\alpha \Phi) (\gamma_\alpha \Phi), \end{aligned} \quad (34)$$

where

$$\bar{\Phi} = (\Phi_1, \Phi_2) \gamma^0 = i(\Phi_2, -\Phi_1). \quad (35)$$

Now it is easy apply these form of the original system (23)-(25) to the case when the fields $\mathbf{J}, \mathbf{\Omega}_1, \mathbf{\Omega}_2$ parametrize the $E_{8(8)}$ group. This means that the

fields $\mathbf{Q}_\alpha, \mathbf{P}_\alpha, \mathbf{\Phi}$ are in the Lie algebra of $E_{8(8)}$ that is they can be represented as superpositions (with local coefficients) of the generators of this algebra which generators consist of 120 antisymmetric matrices \mathbf{X}^{IJ} ($I, J, K, L = 1, 2, \dots, 16$) and 128 symmetric matrices \mathbf{Y}^A ($A, B, C, D = 1, 2, \dots, 128$) each of the size 248×248 (the concrete realization of these matrices see, for example, in [4]). The defining relations are:

$$[\mathbf{X}^{IJ}, \mathbf{X}^{KL}] = \delta^{IL}\mathbf{X}^{JK} + \delta^{JK}\mathbf{X}^{IL} - \delta^{IK}\mathbf{X}^{JL} - \delta^{JL}\mathbf{X}^{IK} , \quad (36)$$

$$[\mathbf{X}^{IJ}, \mathbf{Y}^A] = -\frac{1}{2}\Gamma_{AB}^{IJ}\mathbf{Y}^B , \quad (37)$$

$$[\mathbf{Y}^A, \mathbf{Y}^B] = \frac{1}{4}\Gamma_{AB}^{IJ}\mathbf{X}^{IJ} , \quad (38)$$

and meaning of the constants Γ_{AB}^{IJ} we explained in Appendix. The Lie algebra values of the fields of interest are⁶:

$$\mathbf{Q}_\alpha = \frac{1}{2}q_\alpha^{IJ}(x^0, x^1)\mathbf{X}^{IJ} , \quad \mathbf{P}_\alpha = p_\alpha^A(x^0, x^1)\mathbf{Y}^A , \quad \mathbf{\Phi} = \psi^A(x^0, x^1)\mathbf{Y}^A . \quad (39)$$

The physical degrees of freedom in the fields $q_\alpha^{IJ}(x^0, x^1)$ and $p_\alpha^A(x^0, x^1)$ are those produced by 128 space-time physical scalars $\varphi^A(x^0, x^1)$ which parametrize the $E_{8(8)}$ group values of the orthonormal frame $\mathbf{V} = \exp(a^{IJ}\mathbf{X}^{IJ} + \varphi^A\mathbf{Y}^A)$. Here the 120 components $a^{IJ}(x^0, x^1)$ are pure gauge objects which can be chosen in any desirable form by an appropriate orthogonal rotation of the frame \mathbf{V} (for example a^{IJ} can be eliminated, in which case the frame becomes a symmetric matrix). Then all physical degrees of freedom in the frame current (29) come from the scalar fields $\varphi^A(x^0, x^1)$. The each odd coefficient $\psi^A(x^0, x^1)$ (for each fixed value of the index A) in $\mathbf{\Phi}$ represents the two-component Dirac spinor in the two-dimensional space-time x^0, x^1 . In this way the numbers of the physical degrees of freedom (that is the numbers of the arbitrary initial data) for scalars and spinors are the same (256 for scalars since equations of motion for φ^A are of the second order in time and 256 for spinors ψ^A because they satisfy Dirac-like equations of the first order in time but have twice more components). Consequently one can expect that in some way a supersymmetry may be hidden in this model and, as was shown in [1], this is really the case.

Substituting decomposition (39) into equations (33)-(34) and taking into account the relations:

$$\psi^A = \begin{pmatrix} \psi_1^A \\ \psi_2^A \end{pmatrix} , \quad \bar{\psi}^A = i(\psi_2^A, -\psi_1^A) , \quad \bar{\mathbf{\Phi}} = \bar{\psi}^A\mathbf{Y}^A , \quad \bar{\psi}^A\gamma^\alpha\psi^B = -\bar{\psi}^B\gamma^\alpha\psi^A , \quad (40)$$

together with defining commutations (36)-(38), we obtain:

$$\frac{1}{\alpha}\eta^{\alpha\beta} \left[(\alpha p_\alpha^A)_{,\beta} + \frac{1}{4}q_\alpha^{IJ}\Gamma_{AB}^{IJ}\alpha p_\beta^B \right] = -\frac{1}{128}\Gamma_{AB}^{IJ}p_\alpha^B\bar{\psi}^C\gamma^\alpha\Gamma_{CD}^{IJ}\psi^D , \quad (41)$$

⁶The metric \mathbf{J} is symmetric matrix. Then we restrict the matrices $\mathbf{J}\Omega_1, \mathbf{J}\Omega_2$ and $\mathbf{J}\mathbf{H}$ also to be symmetric. Therefore, from (27)-(28) and (32) follows that $\mathbf{\Phi}$ should be symmetric, that is should be a superposition of the symmetric generators \mathbf{Y}^A .

$$\frac{1}{\sqrt{\alpha}}\gamma^\alpha \left[(\sqrt{\alpha}\psi^A)_{,\alpha} + \frac{1}{4}q_\alpha^{IJ}\Gamma_{AB}^{IJ}\gamma^\alpha\sqrt{\alpha}\psi^B \right] = -\frac{1}{256}\gamma^\alpha\Gamma_{AB}^{IJ}\psi^B\bar{\psi}^C\gamma_\alpha\Gamma_{CD}^{IJ}\psi^D. \quad (42)$$

These equations are similar to those in the Nicolai $N = 16$ model but expressions (41)-(42) do not use the dotted spinor and dotted Γ -coefficients. To get an exact coincidence with paper [1] it is necessary to pass from the spinor ψ^A to the spinor $\chi^{\dot{A}}$ ($\dot{A}, \dot{B}, \dot{C}, \dot{D} = 1, 2, \dots, 128$) and from the coefficients Γ_{AB}^{IJ} to $\Gamma_{\dot{A}\dot{B}}^{IJ}$ in accordance with transformations:

$$\psi^A = 2\chi^{\dot{A}}(\mathbf{C})_{\dot{A}A}, \quad \Gamma_{AB}^{IJ} = (\mathbf{C}^{-1})_{A\dot{A}}\Gamma_{\dot{A}\dot{B}}^{IJ}(\mathbf{C})_{\dot{B}B} \quad (43)$$

where the 128×128 matrix \mathbf{C} [with entries $(\mathbf{C})_{\dot{A}A}$] and inverse to it \mathbf{C}^{-1} [with entries $(\mathbf{C}^{-1})_{A\dot{A}}$] satisfy the relation:

$$\tilde{\mathbf{C}} = -4i\mathbf{C}^{-1}. \quad (44)$$

This matrix can be represented as $\mathbf{C} = \sqrt{-4i}\mathbf{O}$, where the orthogonal matrix \mathbf{O} , acting in the internal spinor representation space, corresponds to the improper rotation (including reflections)⁷. After transformations (43)-(44) equations (41)-(42) take the following form:

$$\eta^{\alpha\beta} \left[\frac{1}{\alpha}(\alpha p_\alpha^A)_{,\beta} + \frac{1}{4}q_\alpha^{IJ}\Gamma_{AB}^{IJ}p_\beta^B \right] = \frac{i}{8}\Gamma_{AB}^{IJ}p_\alpha^B\bar{\chi}^{\dot{C}}\gamma^\alpha\Gamma_{\dot{C}\dot{D}}^{IJ}\chi^{\dot{D}}, \quad (45)$$

$$-i\gamma^\alpha\frac{1}{\sqrt{\alpha}} \left[(\sqrt{\alpha}\chi^{\dot{A}})_{,\alpha} + \frac{1}{4}q_\alpha^{IJ}\Gamma_{\dot{A}\dot{B}}^{IJ}\sqrt{\alpha}\chi^{\dot{B}} \right] = \frac{1}{16}\gamma^\alpha\Gamma_{\dot{A}\dot{B}}^{IJ}\chi^{\dot{B}}\bar{\chi}^{\dot{C}}\gamma_\alpha\Gamma_{\dot{C}\dot{D}}^{IJ}\chi^{\dot{D}}, \quad (46)$$

which form is the same⁸ as in paper [1].

The (45)-(46) are the basic physical equations of motion for the $N = 16$ supergravity. All other fields in this theory (conformal factor λ^2 and gravitinos) can easily be found in terms of the solutions of the system (45)-(46). To find solutions of these principal equations one should return to the system (23)-(26) and construct some (as simple as possible) background solutions for the fields $\mathbf{J}^{(0)}, \mathbf{\Omega}_1^{(0)}, \mathbf{\Omega}_2^{(0)}, \mathbf{H}^{(0)}$ which parametrize the $E_{8(8)}$ group. Then from (20)

⁷Otherwise we cannot construct the matrix spinor Φ with the same chirality as spinorial generator \mathbf{Y}^A from the the spinors $\chi^{\dot{A}}$ of the opposite chirality. The analogous trick works well in the 4-dimensional special relativity [8] and we hope it can be generalized to the present case. However, the rigorous proof remains to be seen.

⁸In paper [1] there is a misprint with the numerical coefficient in front of the right hand side of equation corresponding to our (45). The correct value for this coefficient (namely $i/8$) have been found by Nicolai and Warner later in paper [5]. The numerical coefficients in front of the right hand side of equation (46) and in corresponding equation in paper [1] both are correct. The difference ($1/16$ and $1/24$) is due to the fact that in the r.h.s. of (46) we used the sum $\gamma_\alpha \dots \gamma^\alpha \dots$ only over $\alpha = 0, 1$ while in [1] this summation was extended up to the addend $\gamma_3 \dots \gamma^3 \dots$, where $\gamma_3 = \gamma^3 = \gamma_0\gamma_1$. However, the difference manifests itself only in the aforementioned overall factors.

In our paper we don't use the special designation D_α for covariant (with respect to the local orthogonal rotations of the frame \mathbf{V}) derivatives. In all our equations such derivatives are written in the explicit forms.

follows the background matrix $\Psi^{(0)}(\zeta, \eta, \theta^1, \theta^2)$, using which one can integrate the Lax pair (16) to get the corresponding seed solution for the spectral matrix $\hat{\Psi}^{(0)}(\zeta, \eta, \theta^1, \theta^2, s)$. All subsequent steps are closely analogous to the procedure described at the end of the previous section. The only difference is that now some part of the algebraic operations should be carried out following the super-mathematical rules. After adding to the background a number of solitons we will obtain the final solutions for the n -solitonic $E_{8(8)}$ fields $\mathbf{J}(\zeta, \eta)$, $\mathbf{\Omega}_1(\zeta, \eta)$, $\mathbf{\Omega}_2(\zeta, \eta)$ and we can choose any orthonormal frame $\mathbf{V}(\zeta, \eta)$ satisfying the relation (27) (if necessary, we can pass to any other frame by the corresponding orthogonal rotation). This frame will give the matrices \mathbf{Q}_α and \mathbf{P}_α as is prescribed by the relation (29) and from (39) we will obtain the coefficients $q_\alpha^{IJ}(x^0, x^1)$ and $p_\alpha^A(x^0, x^1)$, namely those appearing in the equations (45)-(46). Also with the help of this frame matrix we will extract from $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ in accordance with formulas (28) and these matrices will give us the matrix spinor Φ (32). Then the 128 spinors of interest $\chi^{\dot{A}}$ will follow from the third relation (39) and first relation (43). Calculated in this way quantities $q_\alpha^{IJ}(x^0, x^1)$, $p_\alpha^A(x^0, x^1)$, $\chi^{\dot{A}}(x^0, x^1)$ are exactly those which will satisfy automatically equations (45)-(46) of the $N = 16$ supergravity model.

Finally it is necessary to stress that one should not try to find a local supersymmetry directly in the foregoing integrable ansatz because it corresponds to completely fixed supersymmetry gauges. In paper [1] it was showed that after we chose the gauges corresponding to the conformal flat metric $g_{\alpha\beta}(x^0, x^1)$ and to the special form $\psi_\alpha^I = \gamma_\alpha \psi^I$ for the 16 gravitinos $\psi_\alpha^I(x^0, x^1)$ some residual supersymmetry still remains in the system. This residual freedom can be used to eliminate the superpartner to the function $\alpha(x^0, x^1)$ together with superpartner to the spectral parameter $s(x^0, x^1)$ in the Lax pair. That's way we can use (without loss of generality) in the super Lax representation (16) the quantities α and s as the usual even functions. To have a solution for the general form of the two-dimensional $N = 16$ supergravity one should apply to the solution described here the backward supersymmetric transformations which transformations have been described in papers [5] and [7].

4 Appendix

The sixteen 256×256 gamma-matrices Γ^I can be chosen symmetric and block off-diagonal:

$$\Gamma^I = \begin{pmatrix} 0 & \Gamma_{\dot{A}\dot{A}}^I \\ \Gamma_{\dot{A}\dot{A}}^I & 0 \end{pmatrix}, \text{ where } \Gamma_{\dot{A}\dot{A}}^I = \Gamma_{\dot{A}\dot{A}}^I. \quad (47)$$

The Clifford relations $\Gamma^I \Gamma^J + \Gamma^J \Gamma^I = 2\delta^{IJ} \mathbf{I}$ takes the form:

$$\begin{aligned} \sum_{\dot{A}} (\Gamma_{\dot{A}\dot{A}}^I \Gamma_{\dot{B}\dot{A}}^J + \Gamma_{\dot{A}\dot{A}}^J \Gamma_{\dot{B}\dot{A}}^I) &= 2\delta^{IJ} \delta_{\dot{A}\dot{B}}, \\ \sum_A (\Gamma_{A\dot{A}}^I \Gamma_{A\dot{B}}^J + \Gamma_{A\dot{A}}^J \Gamma_{A\dot{B}}^I) &= 2\delta^{IJ} \delta_{\dot{A}\dot{B}}. \end{aligned} \quad (48)$$

Each of these two relations separately does not represent the Clifford algebra. It is known that the real solutions of equations (48) exists and can be constructed, for example, in the way analogous to what has been done in appendix A of the paper [6] for the group $SO(8)$.

The $SO(16)$ generators of spinorial transformation $\mathbf{\Gamma}^{IJ} = \frac{1}{2}(\mathbf{\Gamma}^I \mathbf{\Gamma}^J - \mathbf{\Gamma}^J \mathbf{\Gamma}^I)$ are block diagonal:

$$\mathbf{\Gamma}^{IJ} = \begin{pmatrix} \Gamma_{AB}^{IJ} & 0 \\ 0 & \Gamma_{\dot{A}\dot{B}}^{IJ} \end{pmatrix}, \quad (49)$$

where

$$\begin{aligned} \Gamma_{AB}^{IJ} &= \frac{1}{2} \sum_A (\Gamma_{A\dot{A}}^I \Gamma_{B\dot{A}}^J - \Gamma_{A\dot{A}}^J \Gamma_{B\dot{A}}^I), \\ \Gamma_{\dot{A}\dot{B}}^{IJ} &= \frac{1}{2} \sum_A (\Gamma_{A\dot{A}}^I \Gamma_{A\dot{B}}^J - \Gamma_{A\dot{A}}^J \Gamma_{A\dot{B}}^I). \end{aligned} \quad (50)$$

Both components Γ_{AB}^{IJ} and $\Gamma_{\dot{A}\dot{B}}^{IJ}$ are antisymmetric under interchange of the upper indices I, J as well as lower indices A, B and \dot{A}, \dot{B} .

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